Pisot conjecture and Rauzy fractals

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§1 Pisot conjecture

Let $\mathcal{A} = \{1, 2, \ldots, d\}$ and denote by $\mathcal{A}^*$ the set of all finite words over $\mathcal{A}$.

A substitution (over $\mathcal{A}$) is a mapping $\sigma : \mathcal{A} \rightarrow \mathcal{A}^*$ and extends to $\sigma : \mathcal{A}^* \rightarrow \mathcal{A}^*$ by requiring $\sigma(w_1w_2) = \sigma(w_1)\sigma(w_2)$.

Example 1 $\mathcal{A} = \{1, 2, 3\}$, $\sigma_R(1) = 12$, $\sigma_R(2) = 13$, $\sigma_R(3) = 1$ (Rauzy substitution)

Let $E_0(\sigma)$ be the associated matrix (abelianization) :

$$E_0(\sigma) = [f(\sigma(1)), f(\sigma(2)), \ldots, f(\sigma(d))]$$

where $f : \mathcal{A}^* \rightarrow \mathbb{Z}^d$ is a homomorphism defined by $f(w_1w_2\cdots w_k) = e_{w_1} + e_{w_2} + \cdots + e_{w_k}$ with $f \circ \sigma = E_0(\sigma) \circ f$. 
A substitution $\sigma$ is **primitive** if there exists $n > 0$ so that for all $i, j \in \mathcal{A}$ the sequence $\sigma^n(i)$ contains $j$. This is equivalent to $E_0(\sigma)^n > 0$.

Throughout this talk, we always assume that $\sigma$ is primitive.

We call $\sigma$ an **irreducible Pisot substitution** if the characteristic polynomial of $E_0(\sigma)$ is irreducible and its Perron-Frobenius root (the maximal eigenvalue) is a Pisot number.
A substitution $\sigma$ naturally induces a mapping on $\mathcal{A}^\mathbb{N}$:

$$\sigma(u) := \sigma(u_0)\sigma(u_1) \cdots \quad \text{if } u = (u_k)_{k \in \mathbb{N}} \in \mathcal{A}^\mathbb{N}.$$ 

Denote by $\mathcal{L}_\sigma$ the set of all the factors of $\sigma^n(a)$ for $\forall a \in \mathcal{A}$ and $\forall n \geq 0$ and also by $\mathcal{L}(x)$ the set of all the factors of $x \in \mathcal{A}^\mathbb{N}$. Define the substitution shift $(X_\sigma, S)$ for $\sigma$ where

$$X_\sigma = \{ x \in \mathcal{A}^\mathbb{N} : \mathcal{L}(x) \subset \mathcal{L}_\sigma \}$$

and $S : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ is the shift: $(Sx)_i = x_{i+1}$ for $x = (x_i)$. If $\sigma$ is primitive, then it is known that $\sigma : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ has at least one periodic point: $\sigma^k(u) = u$ for some $k$. 

2
WE STUDY SHIFT-RECURRENCE PROPERTIES OF $u$.

Since $X_\sigma = X_{\sigma^n}$ for all $n \geq 1$ and since any iteration of a Pisot substitution is also Pisot, we may assume from the beginning that any Pisot substitution $\sigma$ has a fixed point $u$. And if $\sigma$ has a fixed point, then $X_\sigma = \overline{\mathcal{O}(u)}$ where $\overline{\mathcal{O}(u)}$ is the orbit closure of $u$

$$\overline{\mathcal{O}(u)} = \{S^n u : n \in \mathbb{N}\}$$

with respect to some distance.

This $(\overline{\mathcal{O}(u)}, S)$ is known to have a unique ergodic measure $\nu$ and be minimal. $(\overline{\mathcal{O}(u)}, S, \nu)$ is called the substitution dynamical system associated with $\sigma$. 

3
The spectral type of a substitution dynamical system can vary from the weakly mixing one (Dekking-Keane 1978) to the one with discrete spectrum.

**Pisot conjecture**

If $\sigma$ is an irreducible Pisot substitution, then the substitution dynamical system $(\overline{O(u)}, S, \nu)$ has discrete spectrum.

In general, a measure-preserving system $(X, T, \mu)$ is said to have **discrete spectrum** if the eigenfunctions of the unitary operator $U_T f(x) = f(Tx)$ forms a complete orthonormal basis in $L^2(X, \mu)$.

If $(X, T, \mu)$ has discrete spectrum, then $(X, T, \mu)$ can be isomorphic to a rotation on a compact abelian group (classification).

- $\# A = 2 \Rightarrow$ O.K. [Barge-Diamond, Hollander-Solomyak]

- $\# A \geq 3 \Rightarrow$ specific examples [Rauzy, …]
Rauzy was the first who was able to show that the substitutive dynamical system for $\sigma_R$, "Rauzy substitution", has discrete spectrum.

In current terminology, Rauzy represented the orbit closure of the fixed point of $\sigma_R$

$$u = \lim_{n \to \infty} \sigma^n_R(1) = u_0 u_1 \cdots = 1213121\cdots$$

by "a fractal fundamental domain"

$$X = \{ \pi \sum_{j=0}^{k} e^{u_j} | k = 0, 1, 2, \cdots \}$$

of $\Gamma_0$. Furthermore, he showed that the shift $S$ on $(\overline{O(u)}, S, \nu)$ is commuted by the exchange of subdonmains
Since \((T, X)\) singly covers a minimal translation on 2-torus, it follows that \((\mathcal{O}(u), S, \nu)\) has discrete spectrum.

\[
\begin{array}{ccc}
\mathcal{O}(u) & \xrightarrow{S} & \mathcal{O}(u) \\
\downarrow & & \downarrow \\
X & \xrightarrow{T} & X \\
\text{iso.} \downarrow & & \text{iso.} \downarrow \\
\mathbb{T}^2 & \xrightarrow{\tau} & \mathbb{T}^2
\end{array}
\]
Rauzy’s framework has been generalised by Arnoux-Ito into irreducible, unimodular Pisot substitutions with a general combinatorial condition (*strong coincidence*). Yet the conjugacy between the substitution dynamical system and the toral translation remained unproved. The purpose of this talk is to bypass this gap and provide a proof of Pisot conjecture for irreducible unimodular Pisot substitutions with strong coincidence.

**Main Theorem**  *Pisot conjecture is true for any irreducible, unimodular Pisot substitution with strong coincidence condition.*
§2 Rauzy-Ito-Arnoux framework

Let $\mathcal{P}$ be the contractive invariant subspace for the transpose of $E_0(\sigma)$. Put $d = \#A$.

Let $\pi : \mathbb{R}^d \to \mathcal{P}$ be the projection onto $\mathcal{P}$ along the maximal eigenvector of $E_0(\sigma)$.

Let $\Gamma_0$ be the discrete subgroup generated by translations $\pi(e_i - e_j) \ (1 \leq i < j \leq d)$ acting on $\mathcal{P}$.

Every discrete group of isometries of a metric space is properly discontinuous.

Denote the canonical projection by $p : \mathcal{P} \to \mathcal{P}/\Gamma_0$ and the quotient space $\mathcal{P}/\Gamma_0$ by $\mathbb{T}^{d-1}$.
A compact subset of $\mathcal{P}$

\[
X = \{ \pi(\sum_{j=0}^{k} e_{u_j}) | k = 0, 1, 2, \ldots \}
\]

is called the Rauzy fractal for $\sigma$. Set also

\[
X'_i = \{ \pi(\sum_{l=0}^{k} e_{u_l}) | u_k = i \text{ for some } k \}
\]

and

\[
X_i = \{ \pi(\sum_{l=0}^{k-1} e_{u_l}) | u_k = i \text{ for some } k \}
\]

for $1 \leq i \leq d$. Then

\[
X = \bigcup_{i \in A} X'_i = \bigcup_{i \in A} X_i.
\]
These two partitions are related to each other by a translation

\[ X^{(i)'} - \pi(e_i) = X^{(i)} \quad \text{for } 1 \leq i \leq d. \]

Define the domain exchange transformation \( T : X \to X \) by

\[ T : X'_i \to X_i, \quad T(x) = x - \pi(e_i) \]

and its inverse transformation by

\[ T^{-1} : X_i \to X'_i, \quad T^{-1}(x) = x + \pi(e_i). \]
Theorem 2 (Arnoux-Ito 2001) Let $\sigma$ be an irreducible unimodular Pisot substitution over $A$. If $\sigma$ has a strong coincidence, then there exists a dynamical system $(X, T, \nu)$, the domain exchange system, so that it is measure-theoretically isomorphic to $(\overline{O(u)}, S, \nu)$ and is semi-conjugate to a minimal translation on the torus $(\mathbb{T}^{d-1}, \tau)$. Moreover $(X, T)$ is a finite extension of $(\mathbb{T}^{d-1}, \tau)$.

$$
\begin{array}{ccc}
\overline{O(u)} & S & \overline{O(u)} \\
\phi^{-1} & \downarrow & \phi^{-1} \\
X & T & X \\
p & \downarrow & p \\
\mathbb{T}^{d-1} & \tau & \mathbb{T}^{d-1}
\end{array}
$$
Theorem 3 (Arnoux-Ito,Feng-Furukado-Ito-Wu) Under the same condition as Theorem above,

(1) $v(X_i' \cap X_j') = 0$ and $v(X_i \cap X_j) = 0$ for $i \neq j$
where $v$ means the normalised $(d - 1)$-dimensional Lebesgue measure on $\mathcal{P}$. Furthermore, $v(\partial X^{(i)'}) = v(\partial X^{(i)}) = 0$ for all $i \in A$.

(2) $X$ is interior-dense: $X = \overline{\text{int}X}$.
3 Structure of the Proof of Main Theorem

The domain exchange transformations generate the discrete group. But the converse is NOT true! This fact implies that it is not enough to take the quotient by the discrete group $\Gamma_0$ to see the structure of the substitution dynamical system in Rauzy fractals.

Thus, we construct a discrete flow isomorphic to the domain exchange system instead of the toral translation.

\[
\begin{align*}
\overline{O(u)} & \xrightarrow{S} \overline{O(u)} \\
\phi^{-1}\downarrow & \Downarrow \phi^{-1} \\
\downarrow \phi^{-1}
\end{align*}
\]

\[
\begin{align*}
X & \xrightarrow{T} X \\
iso\downarrow & \downarrow iso.
\end{align*}
\]

\[
\begin{align*}
\tilde{X} & \xrightarrow{\tilde{T}} \tilde{X}
\end{align*}
\]
To obtain such a discrete flow, we use

1. Pseudo-distance on Rauzy fractals

2. Halmos-von Neumann theorem
§4 Pseudo-distance on Rauzy fractals

Remember that a function $d_* : X \times X \to \mathbb{R}$ is called a pseudo-distance if it satisfies that for $\forall x, y, z \in X$

\[(1) \ d_*(x, y) \geq 0, \ (2) \ x = y \Rightarrow d_*(x, y) = 0, \]

\[(3) \ d_*(x, y) = d_*(y, x), \ (4) \ d_*(x, y) \leq d_*(x, z) + d_*(z, y). \]

Notice that if you replace "\(\Rightarrow\)" by "\(\Leftrightarrow\)" in (2), then $d_*$ is called a distance on $X$.

If you have a pseudo-distance $d_*$ on $X$, you can make a metric space from $X$(General topology):
Define an equivalent relation on $X$ by

$$x \sim y \iff d_*(x, y) = 0.$$  

Denote the quotient space by $\tilde{X} = X \setminus \sim$. Then you can define the distance $d$ on $\tilde{X}$ by

$$d([x], [y]) := d_*(x, y) \quad \forall [x], [y] \in \tilde{X}.$$
Typical Example

$\mathbb{T}^2 = [0, 1]^2 / \sim$

How to introduce the distance on 2-torus from a pseudo-distance:

$$d^*(x, y) = \min \{ |x - \gamma(y)| : \gamma \in \Gamma_0 \} \quad (x, y \in [0, 1]^2).$$

where $\Gamma_0 = \langle \gamma_1, \gamma_2 \rangle$ and

$$\gamma_1(a, b) = (a + 1, b), \quad \gamma_2(a, b) = (a, b + 1).$$

$\Rightarrow$

$$d^*(x, y) = 0 \iff x \sim y \text{ (identification on the boundary)}.$$
FOR THE MOMENT, SUPPOSE THAT WE HAVE SUCH A PSEUDO-DISTANCE $d_*$ ON RAUZY FRACTAL $X$ THAT

(P1) (isometry) $d_*(Tx, Ty) = d_*(x, y),$

(P2) $d_*(x, y) = 0 \iff x = y$ or $x, y \in \partial X$ with $x - y \in \Gamma_0.$

(P3) The quotient $\bar{X}$ is compact and $\bar{T} : \bar{X} \to \bar{X}$ by $\bar{T}[x] := [Tx]$ is minimal.
Then, as above, \((\tilde{X}, d)\) is a compact metric space.

Define also \(\tilde{U} : \tilde{X} \to \tilde{X}\) by \(\tilde{U}[x] := [T^{-1}x]\). It is easy to see that \(\tilde{T}\tilde{U} = \tilde{U}\tilde{T} = \text{id}_{\tilde{X}}\). Thus \(\tilde{T}\) is invertible.

We call \((\tilde{X}, \tilde{T})\) the domain exchange flow to emphasize its invertibility.

\((P1)\) implies that \(\tilde{T}\) is an isometry: \(d(\tilde{T}[x], \tilde{T}[y]) = d([x], [y])\).
\(\Rightarrow \) \((\tilde{X}, \tilde{T})\) is a homeomorphism on a compact metric space.
Theorem 4 (Halmos-von Neumann) Let $\tilde{T} : \tilde{X} \to \tilde{X}$ be a homeomorphism on a compact metric space. The followings are equivalent.

(i) $\tilde{T}$ is topologically transitive ($\exists [x_0] \in \tilde{X}$ $O([x_0]) = \{\tilde{T}^n[x_0]\}_{n \in \mathbb{Z}}$ is dense in $\tilde{X}$) and is an isometry for some metric $d$ on $\tilde{X}$

(ii) $\tilde{T}$ is topologically conjugate to a minimal rotation on a compact abelian metric group.

A group structure on $\tilde{X}$ can be introduced through $O([x_0])$: a multiplication $*$ is defined by $\tilde{T}^n[x_0] * \tilde{T}^m[x_0] := \tilde{T}^{n+m}[x_0]$. The isometry $d(\tilde{T}[x], \tilde{T}[y]) = d([x], [y])$ allows group operations to extend on $\tilde{X}$. For any $[x] \in \tilde{X}$, $\tilde{T}[x] = \lim_{n \to \infty} \tilde{T}^n[x_0] = \lim_{n \to \infty} \tilde{T}[x_0] * \tilde{T}^n[x_0] = (\tilde{T}[x_0]) * [x]$, thus a rotation.
(P1),(P2) and (P3) guarantees that $\tilde{T}$ satisfies (i) of Halmos-von Neumann theorem. So $\tilde{T}$ is conjugate to a minimal rotation on a compact abelian group.

Let $G$ be a compact topological group. Then there exists a unique probability measure $m$ on $G$ so that

$$m(xE) = m(E) \quad \forall x \in G \quad \forall E \in \mathcal{B}(G).$$

This measure is called the normalised Haar measure.

The $T$-invariant ergodic measure $\nu$ on $X$ induces a $\tilde{T}$-invariant measure $\mu$ on $\tilde{X}$: $\mu = \nu \circ q^{-1}$ where $q : X \to \tilde{X}$, $q(x) = [x]$.

We can prove

**Lemma 5** The induced invariant measure $\mu = \nu \circ q^{-1}$ is the normalised Haar measure on $\tilde{X}$. 

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(P2) implies that the difference between $\bar{X}$ and $X$ occurs only on the boundary $\partial X$ and the interior $\text{int}X$ is preserved by $q$.

Since $v(\partial X) = 0$, we conclude

**Lemma 6** $(\bar{X}, \tilde{T}, \mu)$ is measure-theoretically isomorphic to $(X, T, v)$.

\[
\begin{array}{ccc}
(X, v) & \xrightarrow{T} & (X, v) \\
\downarrow q & & \downarrow q \\
(\bar{X}, \mu) & \xrightarrow{\tilde{T}} & (\bar{X}, \mu)
\end{array}
\]
We refer to the following two standard theorems in Ergodic theory.

**Theorem 7** Let $G$ be a compact group and $\tilde{T}(g) = ag$ be a rotation of $G$. Then $\tilde{T}$ is ergodic w.r.t the normalised Haar measure iff $\{a^n\}_{n \in \mathbb{Z}}$ is dense in $G$.

**Theorem 8** Let $\tilde{T}(g) = ag$ be an ergodic rotation of a compact abelian group $G$. Then $\tilde{T}$ has discrete spectrum.
Consequently, $(\tilde{X}, \tilde{T}, \mu)$ has discrete spectrum and so do $(X, T, \nu)$ and $(\mathcal{O}(\mu), S, \nu)$.

Which proves Pisot conjecture for irreducible, unimodular Pisot substitutions with strong coincidence.
§5 How to construct such a nice pseudo-distance

WE HOPE TO CONSTRUCT A SIMILAR PSEUDO-DISTANCE ON RAUZY FRACTALS THAT IDENTIFIES THE BOUNDARY JUST AS WE SEE IN THE CASE OF 2-TORUS.

Difficulties:

- How to identify the fractal boundary?
- No guarantee that $X$ is the closure of a fundamental domain.
Following Feng-Furukado-Ito-Wu, we describe the boundary of Rauzy fractals by a local tiling.

**Theorem 9 (FFIW,etc)** Given $i \in A$ and $n \geq 1$, the following set equation holds:

$$X'_{i} = \sum_{j \in A} \sum_{\sigma^n(j) = P_{k,n}^{(j)} i S_{k,n}^{(j)}} E_0(\sigma)^n X'_j - \pi(f(S_{k,n}^{(j)})).$$

Similarly

$$X_{i} = \sum_{j \in A} \sum_{\sigma^n(j) = P_{k,n}^{(j)} i S_{k,n}^{(j)}} E_0(\sigma)^n X_j + \pi(f(P_{k,n}^{(j)})).$$

Cover a given open set by subtiles including (a copy of) the center pieces.
§6 Boundary identification

Definition 1 We say that $X_i'$ and $X_j'$ have contact face at $x$ if

$$x \in \text{int}X \cap X_i' \cap X_j' \text{ and } v(X_i' \cap X_j') = 0.$$  

Proposition 10 Suppose that $X_i'$ and $X_j'$ have contact face at $x$. Then neither $X_i$ nor $X_j$ has contact face at $T(x)$ with any $X_k$. In particular, $T(X_i' \cap X_j') \subset \partial X$ for $i \neq j$.

Each contact face reflects a certain structure in the fixed point $u$. 
For example, in Rauzy substitution case, $X'_1$ and $X'_2$ have contact face.

$$u_{[0,2]} = 121(3121\ldots) \in X'_1, \quad u_{[0,8]} = 121312112(1312\ldots) \in X'_2$$

$$\sigma^3(2) = 12\overline{3121}$$

$$\sigma^3(1) = 121\overline{3121}$$

The word of difference $u_{[3,8]} = 312112$ is the same as $\sigma^3(2)$ in terms of $\pi$.

I don't want to show the exact formulation here, but each contact face reflects that kind of structure in the symbolic sequence. In turn, this structure in the symbolic sequence regulates the formation of contact face!
Definition 2 We say that $x, y \in \partial X$ can be 'dynamically' identified if we can find such a pair of subiles $X'_i, X'_j$ with contact face at $a$ that

$$x, y \in T^m a \quad \text{for some } n > 0.$$ 

We can introduce a finite partition of $\partial X$ through a local tiling. If $x, y \in \partial X$ can be 'dynamically' identified, then

$$x - y \in \Gamma_0.$$
§7 Fundamental domain?

**Lemma 11 (Ito-Arnoux)** There exists an integer $l \geq 1$ such that the number of preimages $p^{-1}(\cdot)$ is $l$ a.e.

We see $X$ as "$l$-tiles of a fundamental domain".

NOTICE THAT PISOT CONJECTURE IS EQUIVALENT TO $l = 1$.

Might introduce a pseudo-distance just as in differential geometry, but WE DO NOT KNOW that $\tilde{X}$ is even arcwise-connected.
§4 How to connect $x$ and $y$ in $X$

Let

$$[x, \gamma(y)]_t := x + t(\gamma(y) - x) \quad (0 \leq t \leq 1).$$

Let $C(t)$ be a piecewise smooth curve in $\mathcal{P}$ parametrized by $t \in [0, 1]$. The lift $p^{-1}p(C(t))$ consists of $\gamma(C(t))$ for $\gamma \in \Gamma_0$. Every pair, $\gamma_1(C(t))$ and $\gamma_2(C(t))$, translates to each other by $\Gamma_0$.

**Proposition 12** After a finite number of special translations of $\Gamma_0$ (boundary identification), any point $\xi$ of $[x, \gamma(y)]_t$ enters into $X$. In particular, $\gamma(y)$ is translated into $\gamma' \circ \gamma(y)$ for some $\gamma' \in \Gamma_0$ (folding process).
Definition 3 We say that $x, y \in X$ are **joined** in $X$ if we can connect $x$ and $y$ through the boundary identification.

Definition 4 For $x, y \in X$, define

$$ d_*(x, y) = \inf \{|x - \gamma(y)|\} $$

where infimum is taken over any $\gamma \in \Gamma_0$ by which $x$ and $y$ are joined in $X$.

Remark 1 Definitely $x$ and $y$ are joined by $[x, y]_t$ with $\gamma = \text{id}$. So we can take infimum.
Lemma 13 \( d_\star \) is a pseudo-distance on \( \tilde{X} \).

Remark 2 If \( \# p^{-1}(\cdot) = 1 \) a.e., which we are aiming at, then the definition above coincides with the one for the standard pseudo-distance \( d^\star \).

Our pseudo-distance \( d_\star \) is locally Euclidean.

Proposition 14 Suppose \( x \in \text{int} \tilde{X} \). Then there exists \( \epsilon_0 > 0 \) so that if \( d_\star(x, y) \leq \epsilon \) for any \( 0 \leq \epsilon < \epsilon_0 \), then \( d_\star(x, y) = |x - y| \). In particular, \( d_\star(x, y) > 0 \) if \( x \neq y \).
In summary, we obtain (P2).

**Proposition 15** \( d_*(x, y) = 0 \) if and only if \( x = y \) or \( x, y \in \partial X \) can be dynamically identified.
We finally verify (P3). We shall show that the domain exchange flow \( (\tilde{X}, \tilde{T}) \) inherits minimality from the substitutive dynamical system \((\mathcal{O}(u), S)\).

**Lemma 16** \( \tilde{X} \) is compact.

[Proof.] Since \((\tilde{X}, d)\) is a metric space, we may verify its sequential compactness.

**Proposition 17** *The domain exchange flow* \( (\tilde{X}, \tilde{T}) \) *is minimal.*

Proof. By [Ito-Arnoux] it is proved that \((X, T, \nu)\) is measure-theoretically isomorphic to \((\mathcal{O}(u), S, \mu)\) via \(\phi\). So there exist \(\Omega_1 \subset \Omega, \nu(\Omega_1) = 1\) and \(X_1 \subset X, \nu(X_1) = 1\) so that \(S\Omega_1 \subset \Omega_1, TX_1 \subset X_1\) and \(\phi : X_1 \rightarrow \Omega_1\) is bijective with \(S \circ \phi = \phi \circ T\). Since \((\mathcal{O}(u), S, \mu)\) is minimal, it follows that the orbit of almost every point of \(X\) is dense. Denote such points of \(X\) by \(Y\). Thus \( (\tilde{X}, \tilde{T}) \) is topologically transitive. Since the domain exchange flow is an isometry, this concludes the proof.