Minimal digit sets for parallel addition in non-standard numeration systems

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Joint work with Edita Pelantová and Milena Svobodová
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Signed-digit representations

Base 10 and digit-set $\{-5, \ldots, 0, \ldots, 5\}$ Cauchy 1840

Base 10 and digit-set $\{-6, \ldots, 0, \ldots, 6\}$ Avizienis 1961

Base 2 and digit-set $\{-1, 0, 1\}$ Chow and Robertson 1978
Signed-digit representations

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Redundancy
Algorithm of Avizienis 1961

Base $\beta = b$, $b \geq 3$ integer, parallel addition on alphabet $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$, $b/2 < a \leq b - 1$.

Input: $x_n \cdots x_m$ and $y_n \cdots y_m$ in $\mathcal{A}^*$, $m \leq n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.

Output: $z_{n+1} \cdots z_m$ in $\mathcal{A}^*$ such that

$$z = x + y = \sum_{i=m}^{n+1} z_i \beta^i.$$

for each $i$ in parallel do

0. $z_i := x_i + y_i$

1. if $z_i \geq a$ then $q_i := 1$, $r_i := z_i - b$
   if $z_i \leq -a$ then $q_i := -1$, $r_i := z_i + b$
   if $-a + 1 \leq z_i \leq a - 1$ then $q_i := 0$, $r_i := z_i$

2. $z_i := q_{i-1} + r_i$
Avizienis

\( \beta = 10 \), digit-set \( \{-6, \ldots, 0, \ldots, 6\} \)

\[
\begin{array}{cccccccc}
x & \mapsto & 2 & 5 & 2 & 5 & 5 & 6 & 0 & 3 \\
y & \mapsto & 5 & 1 & 2 & 2 & 5 & 4 & 0 & 6 & 5 \\
z & \mapsto & 5 & 3 & 7 & 4 & 10 & 9 & 6 & 6 & 8 \\
0 & \mapsto & 1 & \overline{10} \\
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0 & \mapsto & & & & & & & & 1 & \overline{10} \\
z & \mapsto & 5 & 4 & 3 & 3 & 1 & 2 & 3 & 3 & 2 \\
\end{array}
\]

Minimal polynomial of \( \beta \) is \( X - 10 \)

\( 1 \ (\overline{10}) \) is a representation of 0
Algorithm of Chow and Robertson 1978

Base $\beta = b = 2a$, $a \geq 1$, parallel addition on $A = \{-a, \ldots, 0, \ldots, a\}$.

---

**Input:** $x_n \cdots x_m$ and $y_n \cdots y_m$ in $A^*$, $m \leq n$,
$x = \sum_{i=m}^{n} x_i \beta^i$ and $y = \sum_{i=m}^{n} y_i \beta^i$.

**Output:** $z_{n+1} \cdots z_m$ in $A^*$ such that $z = x + y = \sum_{i=m}^{n+1} z_i \beta^i$.

for each $i$ in parallel do

0. $z_i := x_i + y_i$

1. if $a + 1 \leq z_i \leq b$ then $q_i := 1$, $r_i := z_i - b$
   if $-b \leq z_i \leq -a - 1$ then $q_i := -1$, $r_i := z_i + b$
   if $-a + 1 \leq z_i \leq a - 1$ then $q_i := 0$, $r_i := z_i$
   if $z_i = a$ and $z_{i-1} > 0$ then $q_i := 1$, $r_i := -a$
   if $z_i = a$ and $z_{i-1} \leq 0$ then $q_i := 0$, $r_i := a$
   if $z_i = -a$ and $z_{i-1} < 0$ then $q_i := -1$, $r_i := a$
   if $z_i = -a$ and $z_{i-1} \geq 0$ then $q_i := 0$, $r_i := -a$

2. $z_i := q_{i-1} + r_i$
Chow and Robertson (Cauchy)

$\beta = 10$, digit-set \{-5, \ldots, 0, \ldots, 5\}

$$
\begin{array}{cccccccc}
  x & \mapsto & 2 & 5 & \overline{1} & 0 & \overline{3} & 2 & 0 & 3 \\
  y & \mapsto & 1 & 3 & \overline{1} & 2 & 5 & \overline{5} & 3 & 5 & 5 \\
  z & \mapsto & 1 & 5 & \overline{6} & 1 & 5 & \overline{8} & 5 & 5 & 8 \\
  0 & \mapsto & \overline{1} & 10 \\
  0 & \mapsto & \overline{1} & 10 \\
  0 & \mapsto & 1 & \overline{10} \\
  0 & \mapsto & 1 & \overline{10} \\
  0 & \mapsto & 1 & \overline{10} \\
 z & \mapsto & 1 & 4 & 4 & 1 & 4 & 3 & \overline{4} & \overline{4} & \overline{2}
\end{array}
$$
Excursion into symbolic dynamics

A subset $S \subseteq \mathcal{A}^\mathbb{Z}$ is a symbolic dynamical system if it is closed and shift-invariant.

$S \subseteq \mathcal{A}^\mathbb{Z}$ and $T \subseteq \mathcal{B}^\mathbb{Z}$ symbolic dynamical systems.
$\varphi : S \rightarrow T$ is a $p$-local function if $\exists r, t > 0$, and $\exists \Phi : \mathcal{A}^p \rightarrow \mathcal{B}$, with $p = r + t + 1$, such that if $u = (u_i)_{i \in \mathbb{Z}} \in \mathcal{A}^\mathbb{Z}$ and $v = (v_i)_{i \in \mathbb{Z}} \in \mathcal{B}^\mathbb{Z}$, then

$$v = \varphi(u) \iff \forall i \in \mathbb{Z}, \ v_i = \Phi(u_{i+t} \cdots u_{i-r}).$$

The image of $u$ by $\varphi$ is obtained through a sliding window of length $p$.
$r$ is the memory and $t$ is the anticipation of $\varphi$.
$\varphi$ is called a sliding block code.
Excursion into symbolic dynamics

A subset $S \subseteq \mathcal{A}^\mathbb{Z}$ is a **symbolic dynamical system** if it is closed and shift-invariant.

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$r$ is the **memory** and $t$ is the **anticipation** of $\varphi$.  
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A function is computable in parallel iff it is a local function.
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A function is computable in parallel iff it is a local function.  

A local function is computable by a finite sequential transducer.
Differences between the two algorithms

Decision (choice) in step 1:
- Avizienis algorithm is **neighbour free**.
- Chow and Robertson algorithm is **neighbour sensitive**.

Locality: Addition on $\mathcal{A}$ is a function from $(\mathcal{A} + \mathcal{A})^\mathbb{Z}$ to $\mathcal{A}^\mathbb{Z}$
- Avizienis addition is **2-local**.
- Chow and Robertson addition is **3-local**.
Parallel addition

Theorem
(Frougny, Pelantová and Svobodová 2011)
Let $\beta$ with $|\beta| > 1$ be an algebraic number. If all the algebraic conjugates of $\beta$ have modulus $\neq 1$ then one can find an alphabet $A = \{-a, \ldots, 0, \ldots, a\}$ on which parallel addition is possible.
Parallel addition

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The algorithm is a generalization of Avizienis.
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The algorithm is a generalization of Avizienis.

$a$ can be larger than necessary.
Remark
Let $\beta$ with $|\beta| > 1$ be an algebraic number of degree $d$.

- If $d$ is odd or
- if $d = 2$ or
- if $d$ is even $\geq 4$ and the minimal polynomial of $\beta$ is not reciprocal,

then $\beta$ has no conjugate of modulus 1.
The Golden Mean

$$\beta = \frac{1+\sqrt{5}}{2},$$ the Golden Mean.

Every real number $\geq 0$ has an expansion on alphabet $\{0, 1\}$. Addition is not local on $\{0, 1\}$.
The Golden Mean

\[ \beta = \frac{1 + \sqrt{5}}{2}, \] the Golden Mean.
Every real number \( \geq 0 \) has an expansion on alphabet \{0, 1\}.
Addition is not local on \{0, 1\}.

Using \( \beta^4 + \frac{1}{\beta^4} = 7 \), addition on \{-5, \ldots, 5\} is a 9-local function.
The Golden Mean

\[ \beta = \frac{1 + \sqrt{5}}{2}, \text{ the Golden Mean.} \]
Every real number \( \geq 0 \) has an expansion on alphabet \( \{0, 1\} \).
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Using \( \beta^4 + \frac{1}{\beta^4} = 7 \), addition on \( \{-5, \ldots, 5\} \) is a 9-local function.

Using \( \beta^2 + \frac{1}{\beta^2} = 3 \), addition on \( \{-3, \ldots, 3\} \) is a 13-local function.
The Golden Mean

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Using \( \beta^2 + \frac{1}{\beta^2} = 3 \), addition on \( \{-3, \ldots, 3\} \) is a 13-local function.

Addition on \( \{-1, 0, 1\} \) is a 21-local function. The algorithm is neighbour sensitive.
Algorithm A: Base $\beta = \frac{1+\sqrt{5}}{2}$, reduction from $\{-2, -1, 0, 1, 2\}$ to $\{-1, 0, 1, 2\}$.

**Input:** a finite sequence of digits $(z_i)$ of $\{-2, -1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.

**Output:** a finite sequence of digits $(z_i)$ of $\{-1, 0, 1, 2\}$, with $z = \sum z_i \beta^i$.

for each $i$ in parallel do

1. case $\begin{cases} z_i = -2 \\ z_i = -1 \\ z_i = 0 \text{ and } z_{i+2} < 0 \text{ and } z_{i-2} < 0 \end{cases}$ then $q_i := -1$

   else $q_i := 0$

2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$
Algorithm B: Base $\beta = \frac{1+\sqrt{5}}{2}$, reduction from $\{-1,0,1,2\}$ to $\{-1,0,1\}$.

**Input:** a finite sequence of digits $(z_i)$ of $\{-1,0,1,2\}$, with $z = \sum z_i \beta^i$.

**Output:** a finite sequence of digits $(z_i)$ of $\{-1,0,1\}$, with $z = \sum z_i \beta^i$.

for each $i$ in parallel do

1. case

   \[
   \begin{cases}
   z_i = 2 \\
   z_i = 1 \text{ and } (z_{i+2} \geq 1 \text{ or } z_{i-2} \geq 1) \\
   z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 2 \\
   z_i = 0 \text{ and } z_{i+2} = z_{i-2} = 1 \text{ and } z_{i+4} \geq 1 \text{ and } z_{i-4} \geq 1 \\
   z_i = 0 \text{ and } z_{i+2} = 2 \text{ and } z_{i-2} = 1 \text{ and } z_{i-4} \geq 1 \\
   z_i = 0 \text{ and } z_{i-2} = 2 \text{ and } z_{i+2} = 1 \text{ and } z_{i+4} \geq 1
   \end{cases}
   \]

   then $q_i := 1$

2. $z_i := z_i - 3q_i + q_{i+2} + q_{i-2}$

   else $q_i := 0$
Algorithm G: Base $\beta = \frac{1+\sqrt{5}}{2}$, parallel addition on $\mathcal{A} = \{-1, 0, 1\}$.

**Input:** two finite sequences of digits $(x_i)$ and $(y_i)$ of $\{-1, 0, 1\}$, with $x = \sum x_i \beta^i$ and $y = \sum y_i \beta^i$.

**Output:** a finite sequence of digits $(z_i)$ of $\{-1, 0, 1\}$ such that

$$z = x + y = \sum z_i \beta^i.$$

for each $i$ in parallel do

0. $v_i := x_i + y_i$
1. use Algorithm A with input $(v_i)$ and output $(w_i)$
2. use Algorithm B with input $(w_i)$ and output $(z_i)$
Lower bounds of minimal alphabets

A finite alphabet of contiguous integers containing 0 with at least two elements. $\beta$ algebraic number, $|\beta| > 1$

Theorem

1. $\beta$ a real algebraic number $> 1$. If addition on $\mathcal{A}$ is computable in parallel then

$$\# \mathcal{A} \geq \lceil \beta \rceil$$

2. $\beta$ an algebraic integer with minimal polynomial $f(X)$. If addition on $\mathcal{A}$ is computable in parallel then

$$\# \mathcal{A} \geq |f(1)|$$

If $\beta$ is a real algebraic integer $> 1$ then

$$\# \mathcal{A} \geq |f(1)| + 2$$
In the previous theorem

1. “$\# A \geq \lceil \beta \rceil$" can be replaced by

   “$\# A \geq \max\{\lceil \gamma \rceil \mid \gamma \text{ or } \gamma^{-1} \text{ is a positive conjugate of } \beta \}$”.

2. “$\beta$ is an algebraic integer" can be replaced by “$\beta$ or $\frac{1}{\beta}$ is an algebraic integer"

   “$\beta$ is an algebraic integer $> 1$" can be replaced by “$\beta$ is an algebraic integer and one of its algebraic conjugates is $> 1$".
Addition versus conversion

\[ A = \{ m, \ldots, 0 \ldots, M \}. \]

1. \( m = 0 \): Addition on \( A \) is parallelizable \( \iff \)
   
   greatest digit elimination: \( A \cup \{ M + 1 \} \rightarrow A \)

   is parallelizable.

2. \( \{-1, 0, 1\} \subset A \): Addition on \( A \) is parallelizable \( \iff \)
   
   greatest digit elimination: \( A \cup \{ M + 1 \} \rightarrow A \)
   
   and
   
   smallest digit elimination: \( \{ m - 1 \} \cup A \rightarrow A \)
   
   are parallelizable.
How to pass from one alphabet allowing parallel addition to another one of same size

Proposition

For \( K, d \in \mathbb{Z} \), where \( 0 \leq d \leq K - 1 \), denote

\[
A_{-d} = \{-d, \ldots, 0, \ldots, K - 1 - d\}.
\]

Let \( \varphi \) be a \( p \)-local function realizing conversion in base \( \beta \) from \( A_0 \cup \{K\} \) to \( A_0 \). If

\[
\varphi(\omega d \bullet d^\omega) = \omega d \bullet d^\omega \quad \text{and}
\]

\[
\varphi(\omega (K - 1 - d) \bullet (K - 1 - d)^\omega) = \omega (K - 1 - d) \bullet (K - 1 - d)^\omega
\]

then addition is performable in parallel on \( A_{-d} \) as well.
\( \beta = b, \ b \geq 2 \) integer. Minimal polynomial \( f(X) = X - b \).
Lower bound \( |f(1)| + 2 = b + 1 \) is attained.

Parallel addition is feasible on any alphabet of cardinality \( b + 1 \) containing 0, in particular on alphabets \( A = \{0, 1, \ldots, b\} \) and \( A = \{-1, 0, 1, \ldots, b - 1\} \) (folklore).

If \( b \) is even, \( b = 2a \), parallel addition is realizable on the alphabet \( A = \{-a, \ldots, a\} \) of cardinality \( b + 1 \) by the algorithm of Chow and Robertson (Cauchy).
Negative integer base

\[ \beta = -b, \ b \geq 2 \text{ integer}. \]

Every integer has a unique finite representation with digits in \( \{0, 1, \ldots, b - 1\} \) (Grünwald 1885).

Minimal polynomial \( f(X) = X + b \). Lower bound \( |f(1)| = b + 1 \) is attained.

**Theorem**

Let \( \beta = -b \in \mathbb{Z}, \ b \geq 2 \). Any alphabet \( A \) of contiguous integers containing 0 with cardinality \( \#A = b + 1 \) allows parallel addition in base \( \beta = -b \) and this alphabet is minimal in size.
Base $\sqrt[\text{k}]{b}$, $b$ integer, $|b| \geq 2$

Proposition
Let $\beta = \sqrt[\text{k}]{b}$, $b$ in $\mathbb{Z}$, $|b| \geq 2$ and $k \geq 1$ integer. Any alphabet $A$ of contiguous integers containing 0 with cardinality $\#A = b + 1$ allows parallel addition.

Use that $\gamma = \beta^k = b$.

Proposition
If $b$ is in $\mathbb{N}$ the polynomial $X^k - b$ is minimal for $\beta$, thus the cardinality $b + 1$ is minimal.
Complex bases

Penney numeration system (1964): every Gaussian integer has a unique finite expansion in base $\beta = -1 + i$ with digits in \{0, 1\}. Thus it is a canonical numeration system. Example: $3 = 1101$.

Minimal polynomial $f(X) = X^2 + 2X + 2$, and lower bound $= |f(1)| = 5$.

$\beta^4 = -4$. Parallel addition is possible on any alphabet of minimal cardinality 5.
Complex bases

Penney numeration system (1964): every Gaussian integer has a unique finite expansion in base $\beta = -1 + i$ with digits in \{0, 1\}. Thus it is a canonical numeration system. Example: $3 = 1101$.
Minimal polynomial $f(X) = X^2 + 2X + 2$, and lower bound $= |f(1)| = 5$.
$\beta^4 = -4$. Parallel addition is possible on any alphabet of minimal cardinality 5.

Knuth numeration system (1955): $\beta = 2i$ with digits in \{0, $\ldots$, 3\}.
Minimal polynomial $f(X) = X^2 + 4$, and lower bound $= |f(1)| = 5$. Parallel addition is possible on any alphabet of minimal cardinality 5.
Complex bases

Penney numeration system (1964): every Gaussian integer has a unique finite expansion in base $\beta = -1 + i$ with digits in $\{0, 1\}$. Thus it is a canonical numeration system. Example: $3 = 1101$. Minimal polynomial $f(X) = X^2 + 2X + 2$, and lower bound $=|f(1)| = 5$. $\beta^4 = -4$. Parallel addition is possible on any alphabet of minimal cardinality 5.

Knuth numeration system (1955): $\beta = 2i$ with digits in $\{0, \ldots, 3\}$. Minimal polynomial $f(X) = X^2 + 4$, and lower bound $=|f(1)| = 5$. Parallel addition is possible on any alphabet of minimal cardinality 5.

$\beta = i\sqrt{2}$ with digits in $\{0, 1\}$. Minimal polynomial $f(X) = X^2 + 2$, and lower bound $=|f(1)| = 3$. Parallel addition is possible on any alphabet of minimal cardinality 3.
\( \beta \) root of \( X^2 = aX - 1, \ a \geq 3 \)

\( \beta \) is a quadratic Pisot unit.
By the greedy algorithm of Rényi 1957, every positive real has an expansion on the canonical alphabet \( C = \{0, \ldots, a-1\} \).
Uniqueness iff avoids any string of the form \((a-1)(a-2)^k(a-1)\).
If no admissibility condition, then redundancy, which is sufficient.

Minimal polynomial \( f(X) = X^2 - aX + 1 \) and lower bound = \(|f(1)| + 2 = a \)

**Theorem**
\( \beta \) root of \( X^2 = aX - 1, \ a \geq 3 \). Every alphabet of size \( a \) containing 0 allows parallel addition.
\[ \beta \text{ root of } X^2 = aX + 1, \ a \geq 1 \]

\( \beta \) is a quadratic Pisot unit.
Every positive real has an expansion on the canonical alphabet \( C = \{0, \ldots, a\} \).
Uniqueness iff avoids any string of the form \( a1 \).
If no admissibility condition, then redundancy, but it’s not sufficient.

\( a = 1 \): Golden Mean. Minimal alphabet has size 3.

Minimal polynomial \( f(X) = X^2 - aX - 1 \) and lower bound = 
\[ |f(1)| + 2 = a + 2 \]

**Theorem**
\( \beta \text{ root of } X^2 = aX + 1, \ a \geq 1 \). Every alphabet of size \( a + 2 \) containing 0 allows parallel addition.
Positive rational base $\beta = a/b$

By a modification of the Euclidean division algorithm any natural integer has a unique and finite expansion on the alphabet $\{0, \ldots, a - 1\}$ in base $\beta = a/b$ (Akiyama, Frougny and Sakarovitch 2008; Frougny and Klouda 2011).

**Example:** $\beta = 3/2$, then $4 = 21$

If $b \geq 2$, $a/b$ is an algebraic number which is not an algebraic integer, so our lower bound is $\lceil a/b \rceil$, which is not attained.

**Theorem**

In base $\beta = a/b$, with $a$ and $b$ co-prime such that $a > b \geq 1$, the only alphabets of minimal cardinality $a + b$ allowing parallel addition are:

- $\{0, \ldots, a + b - 1\}$ and $\{-a - b + 1, \ldots, 0\}$
- every alphabet of cardinality $a + b$ containing $\{-b, \ldots, 0, \ldots, b\}$. 
Negative rational base $\beta = -a/b$

By a modification of the Euclidean division algorithm any integer has a unique and finite expansion on the alphabet $\{0, \ldots, a-1\}$ in base $\beta = -a/b$ (F. and Klouda 2011). Thus $(-a/b, \{0, \ldots, a-1\})$ forms a canonical numeration system.

If $b \geq 2$, $-a/b$ is a negative algebraic number which is not an algebraic integer, so we have no lower bound.

**Theorem**

In base $\beta = -a/b$, with $a$ and $b$ co-prime such that $a > b \geq 1$, every alphabet of minimal cardinality $a + b$ containing 0 allows parallel addition.
<table>
<thead>
<tr>
<th>Base</th>
<th>Canonical alphabet</th>
<th>Minimal alphabet for parallel addition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b \geq 2$ in $\mathbb{N}$</td>
<td>${0, \ldots, b - 1}$</td>
<td>All alphabets of size $b + 1$</td>
</tr>
<tr>
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</tr>
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<td>$-1 + \iota$</td>
<td>${0, 1}$</td>
<td>All alphabets of size 5</td>
</tr>
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<td>$2\iota$</td>
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</tr>
<tr>
<td>$\iota\sqrt{2}$</td>
<td>${0, 1}$</td>
<td>All alphabets of size 3</td>
</tr>
<tr>
<td>$\beta^2 = a\beta - 1$</td>
<td>${0, \ldots, a - 1}$</td>
<td>All alphabets of size $a$</td>
</tr>
<tr>
<td>$\beta^2 = a\beta + 1$</td>
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